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Asymptotics of multiple crack interactions and prediction of effective modulus

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Abstract

An asymptotic analysis for crack interactions is performed to solve the doubly periodic array of unbridged/bridged cracks under unidirectional tension. The solution leads to a closed-form expression for the effective Young's modulus of the cracked body. It is a function of the geometry of the crack array and the bridging stiffness. A comparison with numerical results obtained previously, shows that the closed-form expression yields accurate results at low to moderate levels of crack density and relative crack length, for both unbridged and bridged cracks.

Keywords: Multiple cracks; Asymptotic analysis; Crack bridging; Crack interaction; Effective modulus; Bridging traction; Fibre-reinforced brittle matrix

1. Introduction

The calculation of the effective elastic properties of media containing multiple cracks has received extensive attention. The significance of studying the elastic behaviour and fracture characteristics of a medium containing multiple cracks or microvoids hardly needs stressing. In fibre-reinforced brittle matrices, such as ceramics and cement, the fibre bridging over multiple cracks is an important mechanism for increasing the toughness of the composites. Some of the studies of media containing multiple cracks are based upon the dilute distribution condition where the interaction among the cracks or microvoids is neglected. Under this assumption, the effective elastic properties can be given in explicit forms. The self-consistent model takes into account the interaction in an indirect way and gives the

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effective properties in implicit forms. A literature review on the effective properties of bodies containing multiple cracks is given by Kachanov (1992). The schemes that are based upon the dilute distribution approximation and which neglect the interaction among cracks are only capable of simulating the actual situation accurately at very low levels of the crack density. On the other hand, if a rigorous solution is sought numerically, the final numerical results, which may be very accurate, are of limited use. Even with numerical methods, it is a tedious or even impossible task to deal with a body containing countless cracks or microvoids, not to mention that the density of cracks is most likely to increase with load. Therefore, one simple way to model a body containing multiple cracks is to assume that the cracks are arranged in a regular pattern so that some approximate, but accurate analytical expressions can be obtained to estimate the overall behaviour of the body.

Most studies so far have been devoted to modelling a body containing doubly periodic array of unbridged cracks in two dimensions. For unbridged cracks, Delameter et al. (1975) replaced the cracks by a distribution of dislocations and calculated the stress intensity factor at the crack tips and the effective elastic constants in modes I and II. Karihaloo (1978) solved a more complicated problem involving plastic relaxation zones around the crack tips in three loading modes for rectangular and diamond-shaped arrays of cracks. Horii and Sahasakmontri (1990) revealed an anomaly in the use of the double infinite series which appeared in the works of Delameter et al. (1975) and Karihaloo (1978). Deng and Nemat-Nasser (1992) obtained exact closed-form expressions for crack opening displacements under all three loading modes for a row of periodic cracks in isotropic and transversely isotropic media. These solutions were then used to estimate the overall elastic moduli of a body containing a doubly periodic array of cracks. In their procedure, the condition of dilute distribution is assumed, so the interaction among the crack rows is neglected. The overall moduli are given by explicit closed-form formulas. Regarding the bridged cracks (Karihaloo et al., 1996), using the pseudo-traction technique (Horii and Nemat-Nasser, 1985; Hu et al., 1994) and a superposition procedure, solved the doubly periodic array of bridged cracks. They calculated the mode I stress intensity factor at the cracks tips and the overall Young's modulus, and also simulated the tension-softening process in short fibre-reinforced cementitious composite materials. Their results for the mode I stress intensity factor were shown to be very accurate. The accuracy of the scheme is recapitulated by Karihaloo and Wang (1997). These results, although highly accurate, are only available in a numerical form.

The aim of this paper is to obtain an explicit closed-form expression for the overall Young's modulus of a body containing a doubly periodic array of unbridged or bridged cracks under unidirectional tension. To this end, an asymptotic analysis is performed based upon the exact traction consistency condition on the crack faces. The asymptotic analysis includes the interaction among the crack rows up to a certain degree of accuracy. The overall Young's modulus of the cracked body obtained from the asymptotic analysis is compared with the numerical result obtained from the scheme by Karihaloo et al. (1996). It is shown that the explicit closed-form asymptotic expression yields very accurate results for both unbridged and bridged cracks.

2. Asymptotic analysis

As pointed out above, the overall behaviour of a body containing multiple cracks or voids is generally simulated by assuming the cracks or voids are arranged in a regular pattern. For two-dimensional cases, such a pattern is a doubly periodic array, as shown in Fig. 1. Here, it is assumed that the body is remotely subjected to a stress σ^0 normal to the cracks. Following Horii and Nemat-Nasser (1985) and Karihaloo et al. (1996), the stress σ^0 is regarded as a homogeneous stress in the same body if all the cracks were absent. If the cracks are bridged, their faces are subjected to a normal bridging traction p_n.

Following the superposition procedure of Karihaloo et al. (1996), and the pseudo-traction technique

(Horii and Nemat-Nasser, 1985; Hu et al., 1994), the original problem can be decomposed into a sequence of subsidiary problems in each of which only one row of collinear cracks is considered. In terms of the concept of pseudo-traction, suppose that the faces of each crack is subjected to a distributed pseudo-traction $\sigma^p(x)$; $x \in (-a, +a)$. By superposition of the subsidiary problems, the traction consistency condition on each crack in a subsidiary problem can be written as (Karihaloo et al., 1996; Karihaloo and Wang, 1997)

$$\sigma^p(x) - 2\sum_{j=1}^{+\infty}\int_0^a K(x, x^j)\sigma^p(x^j)\,\mathrm{d}x^j + p_n(x) = \sigma^0 \quad x \in [0, a) \tag{1}$$

Due to symmetry, the pseudo-traction is the same on all cracks in the array. The second term in the left-hand side of Eq. (1) represents the interaction among all the cracks in the original doubly periodic array. The integral runs along the length of a crack. The expression for Kx, x^j, was given by Karihaloo and Wang (1997)

$$\begin{aligned} K(x, x^j) = {} & \frac{2}{W}\mathrm{Re}\left\{\frac{\cos(\pi x^j/W)\sqrt{(\sin(\pi a/W))^2-(\sin(\pi x^j/W))^2}}{\left[(\sin(\pi z/W))^2-(\sin(\pi x^j/W))^2\right]\sqrt{1-(\sin(\pi a/W)/\sin(\pi z/W))^2}}\right\} \\ & - \frac{2y}{W}\mathrm{Im}\left\{\frac{\cos(\pi x^j/W)\sqrt{(\sin(\pi a/W))^2-(\sin(\pi x^j/W))^2}}{\left[(\sin(\pi z/W))^2-(\sin(\pi x^j/W))^2\right]^2\left[1-(\sin(\pi a/W)/\sin(\pi z/W))^2\right]}\right. \\ & \times\left[2\frac{\pi}{W}\sin\frac{\pi z}{W}\cos\frac{\pi z}{W}\sqrt{1-(\sin(\pi a/W)/\sin(\pi z/W))^2}\right. \\ & \left.\left. + \frac{\pi}{W}\frac{(\sin(\pi a/W))^2\cos(\pi z/W)\left[(\sin(\pi z/W))^2-(\sin(\pi x^j/W))^2\right]}{(\sin(\pi z/W))^3\sqrt{1-(\sin(\pi a/W)/\sin(\pi z/W))^2}}\right]\right\} \end{aligned} \tag{2}$$

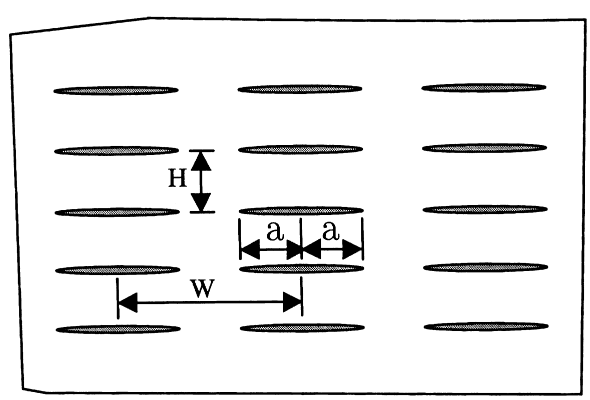

Fig. 1. A body containing doubly periodic array of unbridged/bridged cracks.

where $z = x + iy = x + i(jH)$ $(i = \sqrt{-1},\ j = 1,\ 2, \ldots,\ +\infty)$. jH is the vertical distance of a row of collinear cracks away from the row of cracks on which the traction consistency condition in Eq. (1) is imposed.

As pointed out by Karihaloo and Wang (1997), the kernel $K(x, x^j)$ is not singular. Moreover, $K(x, x^j)$ decays exponentially when j tends to infinity. Thus, the convergence of the sum in Eq. (1) is guaranteed. This enables approximation of the infinite sum in Eq. (1) by a finite number of terms and allows the truncation error to be made as small as desired. Therefore, the integral equation (1) can be easily and accurately solved using Gauss–Legendre quadrature. After solving $\sigma^p(x)$ from Eq. (1), the stress intensity factor and the crack opening displacement can be easily calculated numerically, as can the overall Young's modulus of the cracked body (Karihaloo et al., 1996).

In this paper, we aim to obtain a closed-form solution of the integral equation (1) instead of a numerical one. To this end, we assume that the cracks are so distributed that the higher–order terms (in comparison with terms of order 1) containing $e^{-l\frac{H}{W}\pi}$ $(l \geq 4j)$, $e^{-n\frac{H}{W}\pi}\sin^m\frac{\pi a}{W}$ and $e^{-n\frac{H}{W}\pi}\sin^m\frac{\pi x}{W}$ $(n \geq 2j$ and $m \geq 2)$ either singly or in any combination in $K(x, x^j)$ (Eq. (2)) can be neglected. The asymptotic expression for $K(x, x^j)$ is

$$K_{\mathrm{asym}}(x, x^j) = -\frac{8}{W}e^{-2j\frac{H}{W}\pi}\left[1 + 2j\frac{H}{W}\pi\right]\cos\frac{\pi x^j}{W}\sqrt{\sin^2\frac{\pi a}{W} - \sin^2\frac{\pi x^j}{W}} \tag{3}$$

Substituting Eq. (3) into Eq. (1) gives

$$\sigma^p(x) - 2\sum_{j=1}^{+\infty}\int_0^a K_{\mathrm{asym}}(x, x^j)\sigma^p(x^j)\,dx^j + p_n(x) = \sigma^0 \tag{4}$$

First, we consider the case when the cracks are not bridged, i.e. $p_n(x) = 0$, whence Eq. (4) becomes

$$\sigma^p(x) - 2\sum_{j=1}^{+\infty}\int_0^a K_{\mathrm{asym}}(x, x^j)\sigma^p(x^j)\,dx^j = \sigma^0 \tag{5}$$

As $K_{\mathrm{asym}}(x, x^j)$ does not contain the variable x, the pseudo-traction $\sigma^p(x)$ which satisfies Eq. (5) must be independent of x, that is, $\sigma^p(x)$ must be constant on the crack faces. Thus $\sigma^p(x)$ is given by, after carrying out the integration and the summation,

$$\sigma^p(x) = \frac{1}{1+\alpha}\sigma^0 \tag{6}$$

where

$$\alpha = 4\sin^2\frac{\pi a}{W}e^{-2\frac{H}{W}\pi}\left[\frac{1}{1 - e^{-2\frac{H}{W}\pi}} + \frac{2\frac{H}{W}\pi}{\left(1 - e^{-2\frac{H}{W}\pi}\right)^2}\right] \tag{7}$$

On the other hand, when the cracks are bridged, various bridging laws can be chosen in the numerical solution of Eq. (1). In the present study, we choose the simple linear bridging law

$$p_n(x) = kv(x) \tag{8}$$

where k is the bridging stiffness, and $v(x)$ is the half crack opening displacement. Such a linear bridging law occurs, for example, when the material is reinforced with short fibres (Nemat-Nasser and Hori, 1987; Karihaloo et al., 1996).

Encouraged by the solution for the unbridged case, we assume a constant pseudo-traction $\sigma^p(x) = \sigma^p$ over the crack faces. However, a constant pseudo-traction will render the left-hand side of the Eq. (4)

$$\sigma^p(1+\alpha) + \frac{1}{2}k[v](x) \tag{9}$$

where $[v](x)$ is the total crack opening displacement $v^+(x) - v^-(x)$ which is presented in a closed-form in the work of Deng and Nemat-Nasser (1992) for constant tractions on crack faces. It is evident that the traction consistency condition (4) cannot be satisfied as it stands, because the expression (9) is not a constant. For this reason, we approximate the original traction consistency condition (4) in an average sense as follows

$$\frac{1}{2a}\int_{-a}^{+a}\left[\sigma^p(1+\alpha) + \frac{1}{2}k[v](x)\right]\mathrm{d}x = \sigma^0 \tag{10}$$

The integration $\int_{-a}^{+a}[v](x)\,\mathrm{d}x$ has been carried out analytically by Deng and Nemat-Nasser (1992), whence Eq. (10) gives

$$\sigma^p = \frac{1}{1+\alpha+\beta}\sigma^0 \tag{11}$$

where

$$\beta = -\frac{kW^2}{\pi a E'}\ln\left(\cos\frac{\pi a}{W}\right) \tag{12}$$

$E' = E$ for plane-stress, and $E' = E/(1-\nu^2)$ for plane-strain deformation. Expression (11) reduces to Eq. (6) in a natural way when $\beta = 0$. Thus we denote

$$\eta = \frac{1}{1+\alpha+\beta} \tag{13}$$

for both unbridged and bridged cases, so that the constant pseudo-traction in Eq. (11) can be expressed as

$$\sigma^p = \eta\sigma^0 \tag{14}$$

Substitution of α (Eq. (7)) and β (Eq. (12)) into Eq. (13) yields

$$\eta = \left\{1 + 4\sin^2\frac{\pi a}{W}\mathrm{e}^{-2\frac{H}{W}\pi}\left[\frac{1}{1-\mathrm{e}^{-2\frac{H}{W}\pi}} + \frac{2\frac{H}{W}\pi}{\left(1-\mathrm{e}^{-2\frac{H}{W}\pi}\right)^2}\right] - \frac{kW^2}{\pi a E'}\ln\left(\cos\frac{\pi a}{W}\right)\right\}^{-1} \tag{15}$$

We obtained above the pseudo-traction on the crack faces from asymptotic analysis. It is found to be constant and dependent upon the geometry of the crack array. Horii and Sahasakmontri (1990), using the pseudo-traction method earlier proposed by Horii and Nemat-Nasser (1985), presented a first-order approximate solution of the doubly periodic array of unbridged cracks for modes I and II loading conditions. In their work, constant pseudo-tractions were obtained under the first-order approximation. Their pseudo-traction is equivalent to $\sigma^p(x) - \sigma^0$ in the present paper for mode I case. As they used a superposition scheme based upon a solution for a single crack, the constant pseudo-tractions in their work were given in the form of sums of double series for modes I and II, respectively. The double infinite summation for mode I case is not convergent as the sum depends on the order of summation, thus revealing the anomaly in the use of the superposition method based upon the solution for a single crack. On the other hand, the present superposition scheme and asymptotic analysis give a constant pseudo-traction in a concrete and concise form for mode I loading case.

After calculating σ^p from Eq. (11), the half crack opening displacement $v(x)$ and the overall effective Young's modulus $\bar{E}$ can be calculated following the procedure in the work of Deng and Nemat-Nasser (1992),

$$\frac{v(x)E'}{2a\sigma^0} = \eta\frac{\sqrt{2}}{a}\cos\frac{\pi x}{W}\int_x^a \frac{\tan\dfrac{\pi\xi}{W}}{\sqrt{\cos(2\pi x/W) - \cos(2\pi\xi/W)}}\,\mathrm{d}\xi \tag{16}$$

$$\frac{\bar{E}'}{E'} = \left[1 - \eta\frac{4fW^2}{\pi a^2}\ln\left(\cos\frac{\pi a}{W}\right)\right]^{-1} \tag{17}$$

where $f = a^2/(WH)$is the crack density.

For unbridged cracks ($\beta = 0$), the formula (17) reduces to the dilute distribution solution presented by Deng and Nemat-Nasser (1992) when $H/W \to +\infty$ and/or $\sin\frac{\pi a}{W} \to 0$, that is, when the cracks are sparsely distributed. It is seen that as α and β are always positive such that η is less than 1.0, the current asymptotic analysis will predict a higher value of $\bar{E}'/E'$ than the dilute distribution solution. Also, the normalized Young's modulus $\bar{E}'/E'$ now becomes a function of the geometry of the two-dimensional crack array and of the bridging stiffness.

If the cracked body in Fig. 1 is in a plane-stress state of deformation, $\bar{E}'$ is simply the Young's modulus of the material in the direction perpendicular to the crack faces, whence $E' = E$ for an isotropic medium. However, if the cracked body is under a plane-strain state of deformation, $\bar{E}'$ will necessarily involve the Poisson effect. In the sequel, for the purpose of seeking the overall Young's modulus $\bar{E}$, we simply consider the plane-stress state of deformation. In the next section, the overall Young's modulus predicted by the closed-form expression (17) from the asymptotic analysis will be presented and compared with the numerical solution and the dilute concentration approximation.

3. Results

Before presenting the results, we rewrite β (Eq. (12)) as follows

$$\beta = -\frac{\mathscr{A}W^2}{a^2}\ln\left(\cos\frac{\pi a}{W}\right) \tag{18}$$

where the non-dimensional parameter $\mathscr{A}$ is defined as

$$\mathscr{A} = \frac{ka}{\pi E} \tag{19}$$

Using the above expression for $\mathscr{A}$, Eq. (15) becomes

$$\eta = \left\{ 1 + 4\sin^2\frac{\pi a}{W} e^{-2\frac{H}{W}\pi} \left[\frac{1}{1 - e^{-2\frac{H}{W}\pi}} + \frac{2\frac{H}{W}\pi}{\left(1 - e^{-2\frac{H}{W}\pi}\right)^2} \right] - \mathscr{A}\left(\frac{W}{a}\right)^2 \ln\left(\cos\frac{\pi a}{W}\right) \right\}^{-1} \tag{20}$$

The value of $\mathscr{A}$ depends on the material, that is, the matrix for a fibre-reinforced composite material, and on the bridging mechanism. Its magnitude may be quite different, say, for short fibre-reinforced cementitious composite materials (Karihaloo et al., 1996) and short fibre- reinforced ceramic composite materials (Nemat-Nasser and Hori, 1987). In the following, we consider two cases with $\mathscr{A} = 0.05$ and 1.0, respectively.

For unbridged cracks, the variations of the normalized Young's modulus $\bar{E}/E$ in Eq. (17) with the crack density parameter $f = a^2/(WH)$ for plane-stress state of deformation are shown in Figs. 2–4 for $H/W = 0.25$, 0.5 and 1.0, respectively. Also shown in these figures are the results obtained from the numerical solution of the integral Eq. (1), the results obtained under dilute distribution assumption where the interaction among the crack rows is neglected, and the results calculated using the formula presented by Ju and Chen (1994) for randomly distributed parallel cracks. It is seen that the results of the present asymptotic analysis taking into account the interaction among the crack rows is very close to the accurate numerical solution for all the cases under consideration. Especially, for $f = a^2/(WH) \leq 0.10$, the closed-form asymptotic solution and the numerical solution are almost identical, whereas the results obtained under dilute distribution assumption are only accurate for very small f or very large H/W. The accuracy of the formula presented by Ju and Chen (1994) cannot be judged. It is obvious that the results obtained by the four procedures will get closer with diminishing interaction among the cracks.

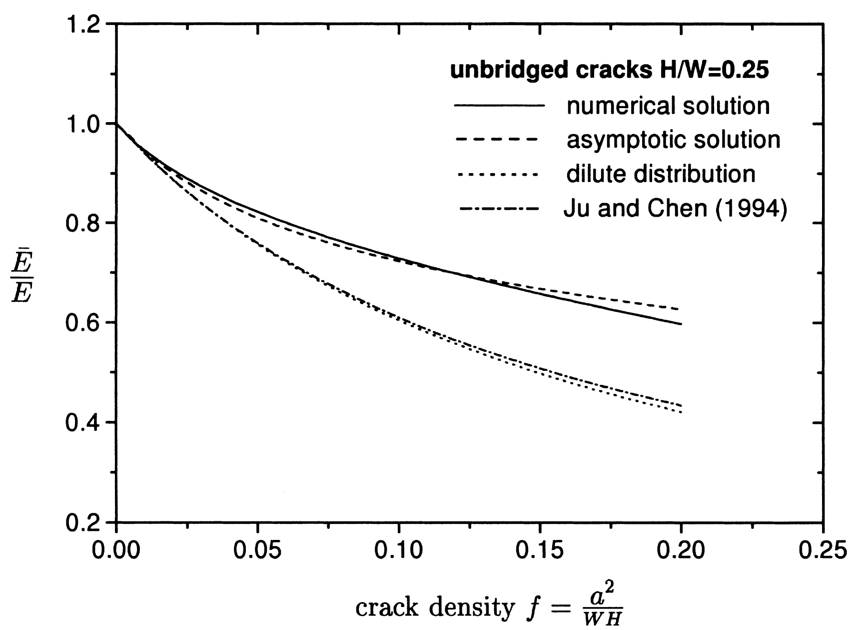

Fig. 2. Variation of normalized modulus with crack density for unbridged cracks when $H/W = 0.25$.

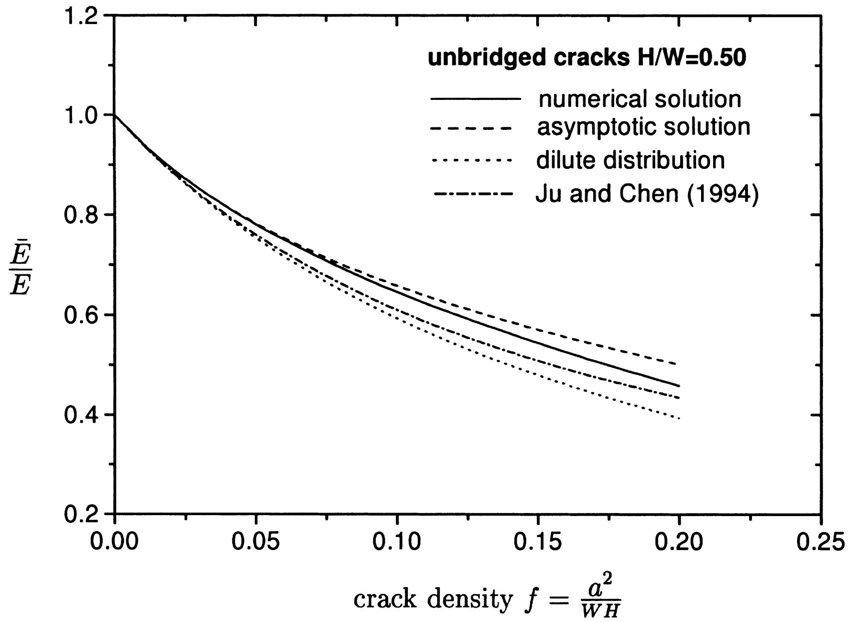

Fig. 3. Variation of normalized modulus with crack density for unbridged cracks when $H/W = 0.50$.

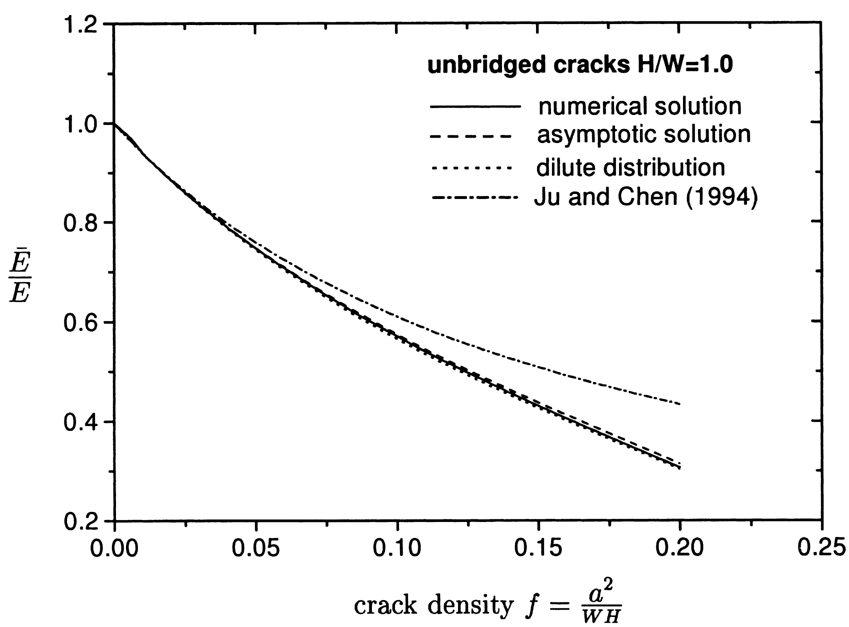

Fig. 4. Variation of normalized modulus with crack density for unbridged cracks when $H/W = 1.0$.

All curves, except that predicted using the formula by Ju and Chen (1994), virtually merge into one in Fig. 4 when $H/W = 1.0$

In the general analytical method for multiple interacting cracks proposed by Kachanov (1987, 1992) the interaction among the cracks is simulated by assuming that the actual non-uniform pseudo-tractions on the crack faces can be replaced by constant uniform tractions. The current asymptotic analysis for the doubly periodic array of unbridged cracks shows that the pseudo-tractions are indeed constant when the higher–order terms containing $e^{-l\frac{H}{W}\pi}$ (for $l \geq 4j$), $e^{-n\frac{H}{W}\pi}\sin^m\frac{\pi a}{W}$ and $e^{-n\frac{H}{W}\pi}\sin^m\frac{\pi x}{W}$ (for $n \geq 2j$ and $m \geq 2$) in $K(x, x^j)$ (Eq. (2)) are negligible. The fast-decaying function $e^{-n\frac{H}{W}\pi}$ dominates the terms $\sin^m\frac{\pi x}{W}$ and $\sin^m\frac{\pi a}{W}$ and determines the accuracy of the asymptotic solution, as is evident from Figs. 2–4. It is also clear from the same figures that the accuracy of the dilute distribution approximation is not controlled so much by the closeness of the cracks in the horizontal direction as it is by the spacing between two neighbouring rows. The dilute distribution approximation yields inaccurate predictions of the overall Young's modulus for $H/W = 0.25$ and $H/W = 0.50$, but it can predict the overall modulus accurately for $H/W \geq 1.0$ for a large range of $2a/W$. For instance, it still works well for $H/W = 1.0$ and $f = 0.20$, when $2a/W = 0.89$, i.e. when the neighbouring tips of two collinear cracks are very close to each other.

In order to examine the relative importance of "collinear" and "stacked" interactions (Kachanov, 1992), let us conduct a sensitivity analysis of the parameter η (Eq. (15)) for unbridged cracks. For this we approximate η by

$$\eta = 1 - 4\sin^2\frac{\pi a}{W}e^{-2\frac{H}{W}\pi} \tag{21}$$

and denote the normalised horizontal and vertical crack spacings by

$$\phi = \frac{2a}{W}, \qquad \psi = \frac{H}{W} \tag{22}$$

The first-order partial derivatives of η with respect to ϕ and ψ are, respectively

$$\frac{\partial\eta}{\partial\phi} = -4\pi\sin\frac{\pi}{2}\phi\cos\frac{\pi}{2}\phi\, e^{-2\pi\psi}, \qquad \frac{\partial\eta}{\partial\psi} = 8\pi\sin^2\frac{\pi}{2}\phi\, e^{-2\pi\psi} \tag{23}$$

whence the ratio of the sensitivities of η to vertical and horizontal crack spacings is

$$\left(\frac{\partial\eta}{\partial\psi}\right)/\left(\frac{\partial\eta}{\partial\phi}\right) = -2\tan\frac{\pi}{2}\phi \tag{24}$$

Thus, for cracks which are densely distributed horizontally (i.e. ϕ is large), the constant pseudo-traction σ^p in Eq. (14) is very sensitive to the vertical spacing between the crack rows. In the works of Kachanov (1987), Kachanov (1992), and of Ju and Chen (1994), the accuracy of the general method proposed by Kachanov (1987) for interactions among multiple cracks is demonstrated on collinear cracks only. Kachanov (1992) pointed out that collinear interaction becomes noticeable only at close horizontal spacings, whereas the stacked interaction has a longer range. This argument, and the above sensitivity analysis suggest that the accuracy of the Kachanov method with respect to the vertical crack separation needs reassessment.

Fig. 5 shows the variation of $\bar{E}/E$ with the crack density for bridged cracks when the non-dimensional bridging stiffness is $\mathscr{A} = 0.05$. Again, the numerical solution represents the result obtained from the numerical solution of the integral Eq. (1), and the asymptotic solution represents that calculated using

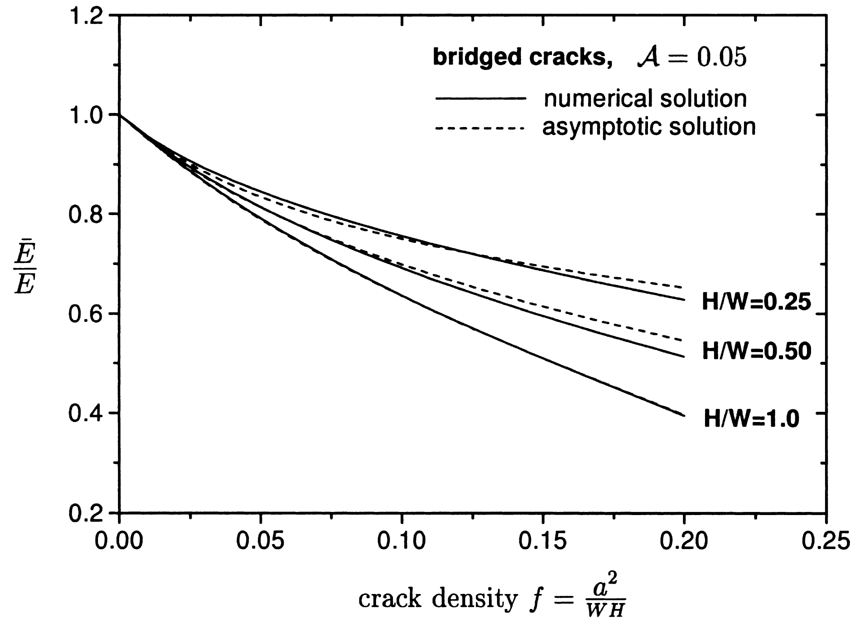

Fig. 5. Variation of normalized modulus with crack density for bridged ($\mathscr{A} = 0.05$) cracks for $H/W = 0.25$, 0.5 and 1.0.

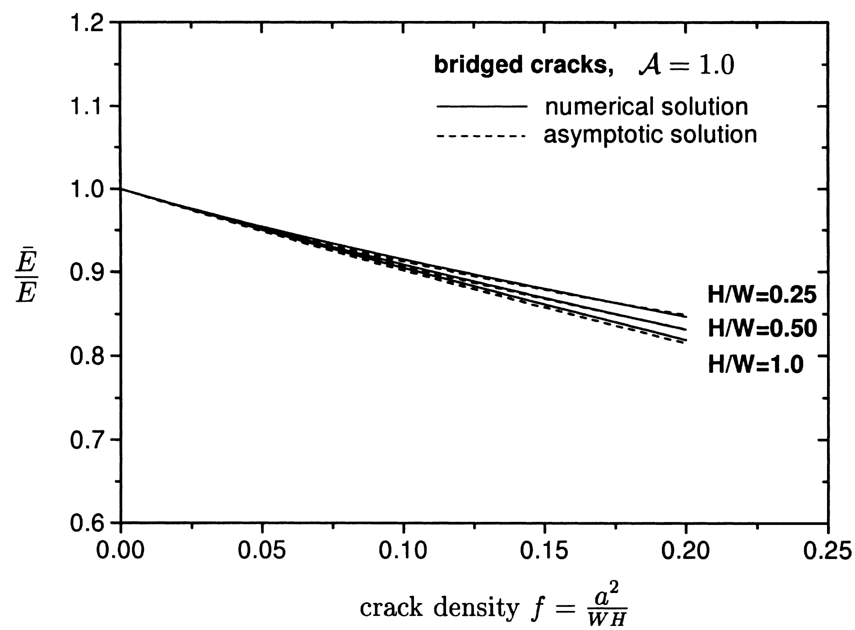

Fig. 6. Variation of normalized modulus with crack density for bridged ($\mathscr{A} = 1.0$) cracks for $H/W = 0.25$, 0.5 and 1.0.

the formula (17). It is seen that for the three values of H/W, the closed-form asymptotic solution gives almost the same result as the numerical solution within the shown range of crack density. Fig. 6 shows the variation of $\bar{E}/E$ with the crack density for $\mathscr{A} = 1.0$, that is, for a strong bridging force. In this case, the effective Young's modulus $\bar{E}$ is only slightly influenced by the density and the geometry of the array of cracks. Again, the result given by the asymptotic solution and that by the numerical solution are basically identical for the considered three values of H/W within the shown range of crack density.

For unbridged doubly periodic array of cracks in an isotropic material, as shown in Fig. 1, Deng and Nemat-Nasser (1992) compared the values of the overall modulus predicted on the basis of dilute approximation, the self-consistent method, and a differential scheme. It was found that under the condition of prescribed far-field stress, the values of the overall modulus predicted by the self-consistent method and the differential scheme were significantly lower than that predicted on the basis of the dilute approximation, except for very small relative crack length $2a/W$. As in the work by Deng and Nemat-Nasser (1992), the variations of $\bar{E}/E$ with the relative crack length $2a/W$ for $H/W = 0.25$, 0.5 and 1.0 are shown in Fig. 7. It is seen from Fig. 7 that except for $H/W = 1.0$ when the curves obtained from the three procedures considered in the present paper virtually merge into one, the dilute distribution approximation gives lower values of $\bar{E}/E$ than the asymptotic formula and the numerical method. This implies that for the doubly periodic array of unbridged cracks, the self-consistent method and the differential scheme which were discussed in the work by Deng and Nemat-Nasser (1992) may considerably underestimate the overall Young's modulus, especially for moderate values of the relative crack length $2a/W$ and low values of H/W.

For the same geometrical parameters of the array of cracks studied in Fig. 7, the variations of the normalized Young's modulus $\bar{E}/E$ predicted by the asymptotic solution and the numerical solution for bridged cracks are shown in Fig. 8. As the non-dimensional bridging stiffness $\mathscr{A}$ defined in Eq. (19) depends on the length of the cracks, it may be assumed, without loss of generality, that $2a/W = 0.5$ when $\mathscr{A} = 0.05$. For other values of $2a/W$, $\mathscr{A} = 0.20(a/W)$. It is seen from Figs. 7 and 8 that $\bar{E}/E$ for

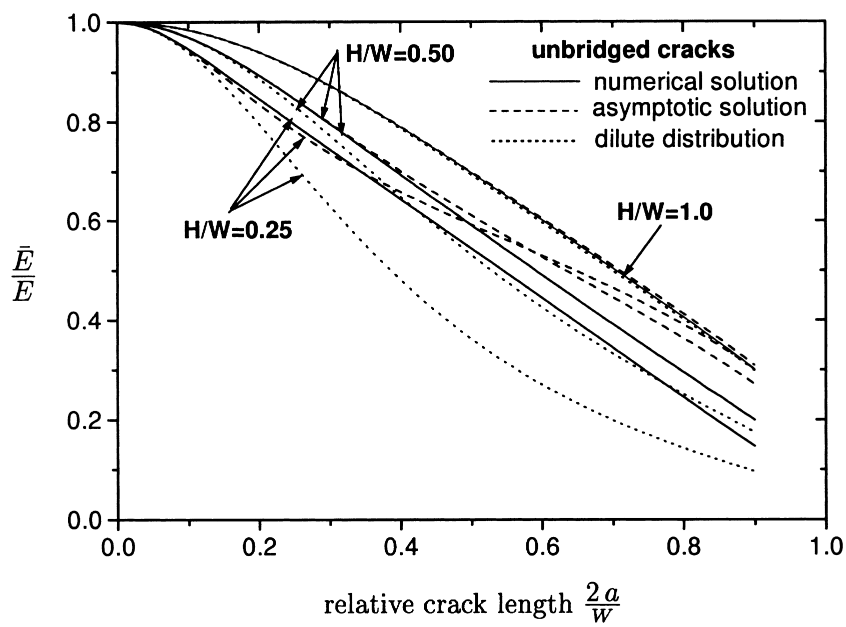

Fig. 7. Variation of normalized modulus with relative crack length for unbridged cracks for $H/W = 0.25$, 0.5 and 1.0.

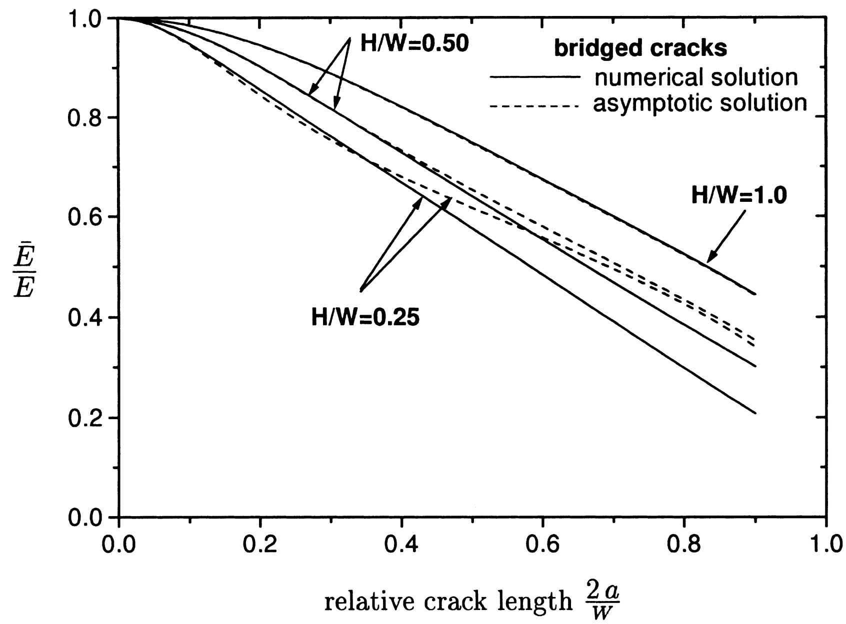

Fig. 8. Variation of normalized modulus with relative crack length for bridged cracks for $H/W = 0.25$, 0.5 and 1.0.

unbridged and bridged cases exhibits the same tendency with the relative crack length $2a/W$. The asymptotic solution can accurately predict the overall modulus as $2a/W < 0.40$ for $H/W = 0.25$ and $H/W = 0.50$, whereas for $H/W \geq 1.0$, the asymptotic solution gives the same results as the numerical solution.

4. Conclusions

An asymptotic analysis is presented in the present paper for the solution of doubly periodic array of unbridged/bridged cracks in an isotropic material under unidirectional tension. The analysis is based on the traction consistency condition on the crack faces and the solution for a single row of collinear periodic cracks. The asymptotic analysis, which includes the interaction among the crack rows, leads to a closed-form expression for the overall Young's modulus of the cracked body. The expression reduces in a natural way to the dilute distribution solution for unbridged cracks available in the literature. For both unbridged and bridged cracks, a comparison of the overall Young's modulus predicted by the asymptotic analysis with that by the accurate numerical solution shows that the asymptotic solution gives very accurate results at low to moderate levels of crack density and relative crack length. The accuracy of the asymptotic solution is more prominent for bridged cracks than for the unbridged ones. It is noted that in real bodies, the distribution of multiple parallel cracks will exhibit some randomness. The possible ranges of variation of the effective moduli of bodies containing randomly distributed multiple parallel cracks are studied by the authors in a separate work (Wang et al., 1999).

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References

Delameter, W.R., Herrmann, G., Barnett, D.M., 1975. Weakening of an elastic solid by a rectangular array of cracks. J. Appl. Mech 42, 74–80.

Deng, H., Nemat-Nasser, S., 1992. Microcrack array in isotropic solids. Mech. Mater 13, 15–36.

Horii, H., Nemat-Nasser, S., 1985. Elastic fields of interacting inhomogeneities. International Journal of Solids and Structures 21, 731–745.

Horii, H., Sahasakmontri, K. 1990. Mechanical properties of cracked solids: validity of the self-consistent method. In: Weng, G.J., Taya, M., Abé, H. (Eds.), Micromechanics and Inhomogeneity — The Toshio Mura 65th Anniversary Volume. Springer-Verlag, New York, pp. 137–159.

Hu, K.X., Chandra, A., Huang, Y., 1994. On interacting bridged-crack systems. International Journal of Solids and Structures 31, 599–611.

Ju, J.W., Chen, T.-M., 1994. Effective elastic moduli of two-dimensional brittle solids with interacting cracks. Part I: Basic formulations. J. Appl. Mech 61, 349–357.

Kachanov, M., 1987. Elastic solids with many cracks: a simple method of analysis. International Journal of Solids and Structures 23, 23–43.

Kachanov, M., 1992. Effective elastic properties of cracked solids: critical review of some basic concepts. Appl. Mech. Rev 45, 304–335.

Karihaloo, B.L., 1978. Fracture characteristics of solids containing doubly-periodic arrays of cracks. Proc. R. Soc. Lond A360, 373–387.

Karihaloo, B.L., Wang, J., Grzybowski, M., 1996. Doubly periodic arrays of bridged cracks and short fibre-reinforced cementitious composites. J. Mech. Phys. Solids 44, 1565–1586.

Karihaloo, B.L., Wang, J., 1997. On the solution of doubly periodic array of cracks. Mech. Mater 26, 209–212.

Nemat-Nasser, S., Hori, M., 1987. Toughening by partial or full bridging of cracks in ceramics and fibre reinforced composites. Mech. Mater 6, 245–269.

Wang, J., Fang, J., Karihaloo, B.L., 1999. Asymptotic bounds on overall moduli of cracked bodies. International Journal of Solids and Structures, submitted.